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HEAT FLUX CONTOURS ON A PLANE FOR PARALLEL RADIATION SPECULARLY REFLECTED FROM A CONE, A HEMISPHERE AND A PARABOLOID

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Abstract— Explicit formulas are derived for the heat flux per unit area on a general receiver surface after planar incident radiation is specularly reflected from a cone, a paraboloid of revolution, and a hemisphere. The formulas for flux density are inverted so that the coordinates of receiver points on the receiver surface are expressed in terms of flux density. These provide closed analytical expressions for flux density contours. Typical results are plotted as contours of constant illumination on a planar receiver surface which is perpendicular to the axis of symmetry of the reflector.

NOMENCLATURE

s_0, energy per unit area per unit time normal to direction of incident radiation. For sunlight it is the solar constant at the location of the reflecting surface;
ρ, reflectivity of reflector;
$\mathcal{E}$, energy flux incident on unit area of receiver;
$\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$, Cartesian unit vectors;
$\boldsymbol{i}$, unit vector which specifies direction of incident radiation;
$\boldsymbol{i}'$, unit vector which specifies direction of reflected radiation;
$\boldsymbol{n}$, outward unit normal to reflecting surface;
μ, angle of incidence on reflecting surface;
x, y, z, Cartesian coordinates of reflecting surface;
r, θ, z, polar cylindrical coordinates of reflecting surface;
$z(r)$, equation of reflecting surface;
$Z(X, Y)$, equation of receiver surface;
f_1, i'_x/i'_z: ratio of x component to z component of unit vector which specifies direction of reflected radiation;
f_2, i'_y/i'_z: ratio of y component to z component of unit vector which specifies direction of reflected radiation;
$J(\theta, R)$, Jacobian determinant connecting $\mathrm{d}x\mathrm{d}y$ of reflector with $\mathrm{d}X\mathrm{d}Y$ of receiver;
I_0, I_1, I_2, Δ, abbreviation for parts of general Jacobian, see equation (2);
α, cone half angle;
β, angle between $\boldsymbol{i}$ and $\boldsymbol{J}$ for radiation incident upon cone;
R_0, base radius of reflector (cone, paraboloid or hemisphere);
$R, r/R_0$, dimensionless reflector radius vector in units of R_0;
d, Z coordinate of intersection of receiving plane normal to z-axis.

I. INTRODUCTION

In [1] a general, analytical formula is derived which gives the flux per unit area on an arbitrary receiving surface for incident radiation specularly reflected from a curved surface. The general results are specialized to the particular form applicable to a reflecting surface with axial symmetry. In this paper we shall apply the latter formula to calculate the flux density contours on a planar receiver surface after plane wave radiation is specularly reflected from a cone, a paraboloid of revolution and a hemisphere. In the case of the cone the formulas will be specialized to apply to any planar receiver, that is, the receiver plane may have any position or orientation with respect to the reflecting cone. The formula for flux density will then be inverted so that one can calculate directly the coordinates of equal flux density on the receiver surface for a specified value of the received flux. This makes possible the direct computation of heat flux contours on the plane. For the paraboloid and hemisphere the receiver plane will be specialized to a plane perpendicular to the symmetry axis of the reflecting surface. The flux flow equation will be inverted in each case so that direct analytical computation of flux contours over any plane perpendicular to the symmetry axis is possible.

The flux density at position X, Y on the surface $Z = Z(X, Y)$ for plane wave radiation which has been specularly reflected by a surface which has axial symmetry about the z-axis, that is, for the surface $z = z(r)$ is given by equation (15) of [1].

$$\mathscr{E} = \frac{s_0 \rho \cos \mu[(\partial z/\partial r)^2 + 1]^{\frac{1}{2}}}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}} |J(r, \theta)|} \tag{1}$$

where

$$J(r, \theta) \equiv [I_0 + (Z - z)I_1 + (Z - z)^2 I_2]/\Delta \tag{2}$$

and

$$I_0 = 1 - (\partial z/\partial r)(f_1 \cos\theta + f_2 \sin\theta)$$

$$I_1 = \cos\theta\left(\frac{\partial f_1}{\partial r} + \frac{1}{r}\frac{\partial f_2}{\partial \theta}\right) + \sin\theta\left(\frac{\partial f_2}{\partial r} - \frac{1}{r}\frac{\partial f_1}{\partial \theta}\right) + \frac{1}{r}\frac{\partial z}{\partial r}\left(f_2\frac{\partial f_1}{\partial \theta} - f_1\frac{\partial f_2}{\partial \theta}\right)$$

$$I_2 = [(\partial f_1/\partial r)(\partial f_2/\partial \theta) - (\partial f_1/\partial \theta)(\partial f_2/\partial r)]/r$$

$$\Delta = 1 - f_1(\partial Z/\partial X) - f_2(\partial Z/\partial Y)$$

$$f_1 \equiv i'_x/i'_z;\ f_2 \equiv i'_y/i'_z.$$

The equation of the line representing the specularly reflected ray in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$, $z = z$) is given by

$$\frac{X - r\cos\theta}{Z(X, Y) - z(r)} = \frac{i'_x(r, \theta)}{i'_z(r, \theta)} \equiv f_1(r, \theta) \tag{3a}$$

$$\frac{Y - r\sin\theta}{Z(X, Y) - z(r)} = \frac{i'_y(r, \theta)}{i'_z(r, \theta)} \equiv f_2(r, \theta). \tag{3b}$$

The direction of the reflected ray $\boldsymbol{i}'$ is given in terms of the normal to the surface S_1 and the direction of the incident ray $\boldsymbol{i}$ by the law of reflection which yields

$$\boldsymbol{i}' = -2\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{i}) + \boldsymbol{i}. \tag{4}$$

The preceding formulas are perfectly general in the sense that the source of radiation may be a plane wave, a point source or extended source by integrating over the sources. s_0 is simply the flux density at the element of reflecting surface whatever the source may be. The form of f_1 and f_2 will be determined by the kind of source, that is whether it is plane wave or point source. f_1 and f_2 also depend on the shape of the reflector surface.

II. SPECULAR REFLECTION FROM CONE TO PLANAR RECEIVER

In order to apply equations (1)–(3) one must specify:

(i) The equation of the reflecting surface which we consider to be the cone $z = r\cot\alpha$ where r is radius of cone measured from z-axis and α is the cone half angle.

(ii) The direction of incident radiation, $\boldsymbol{i}$, which we now consider to be parallel rays from infinity in the xy plane and inclined at an angle β with respect to the y axis:

$$\boldsymbol{i} = -\cos\beta \boldsymbol{J} - \sin\beta \boldsymbol{K} \tag{5}$$

where ($\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$) are the cartesian unit vectors.

(iii) The equation of the receiver surface which we first consider to be a plane normal to the z-axis is $Z = d$ where d is the distance of the plane from the origin. We shall then modify the formulas to allow the receiver surface to be an arbitrary plane given by $Z = D - AX - BY$. A, B and D are constants.

The unit normal to reflecting surface is given by

$$\boldsymbol{n} = \cos\alpha\cos\theta\,\boldsymbol{I} + \cos\alpha\sin\theta\,\boldsymbol{J} - \sin\alpha\,\boldsymbol{K} \quad (6)$$

where (r, θ) are polar coordinates of reflecting point (see Fig. 1). Applying equation (4) along with (5) and (6) to determine the direction of the reflected ray, $\boldsymbol{i}'$, one obtains for $f_1 (\equiv i'_x/i'_z)$ and $f_2 (\equiv i'_y/i'_z)$

$$f_1 = \frac{A\cos\theta\sin\theta - B\cos\theta}{C - D\cos\theta};$$

$$f_2 = \frac{A\sin^2\theta - B\sin\theta - \cos\beta}{C - D\cos\theta} \quad (7)$$

where $A = 2\cos^2\alpha\cos\beta$; $B = \sin(2\alpha)\sin\beta$; $C = -\cos(2\alpha)\sin\beta$; and $D = \sin(2\alpha)\cos\beta$. It should be noted that in the present case both f_1 and f_2 are functions only if the polar angle θ;

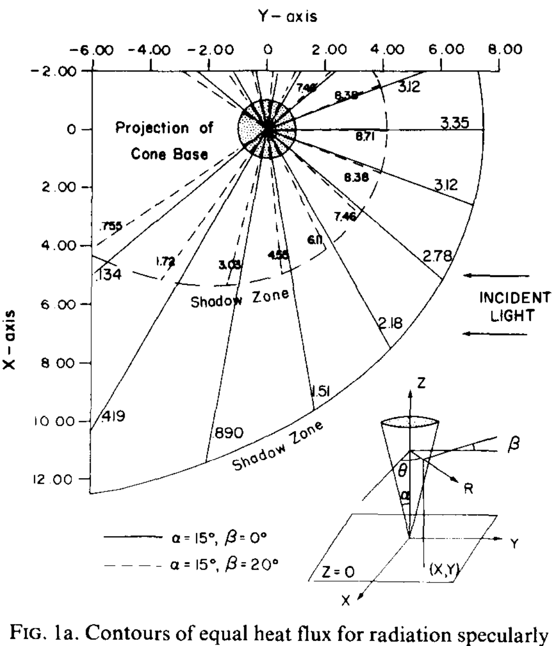

FIG. 1a. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = 0$. Flux values associated with contours represent per cent of incident radiation density in all figures. The reflectivity ρ is taken as 1 for all angles of incidence. The contours are symmetrical with respect to the Z, Y plane in all the figures.

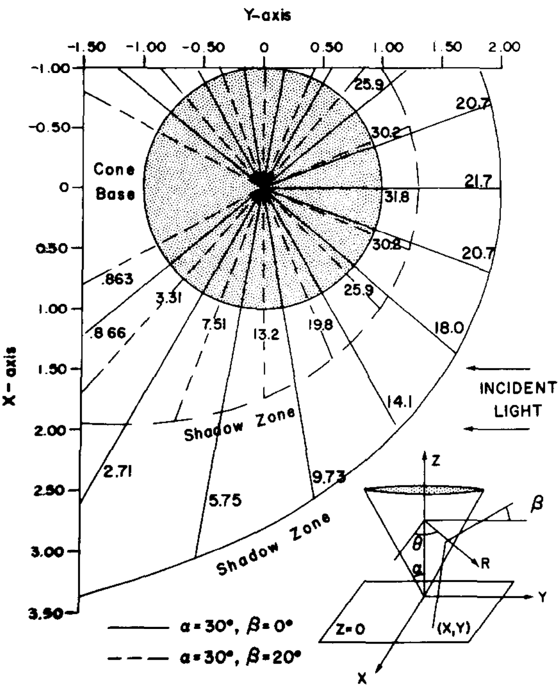

FIG. 1b. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = 0$.

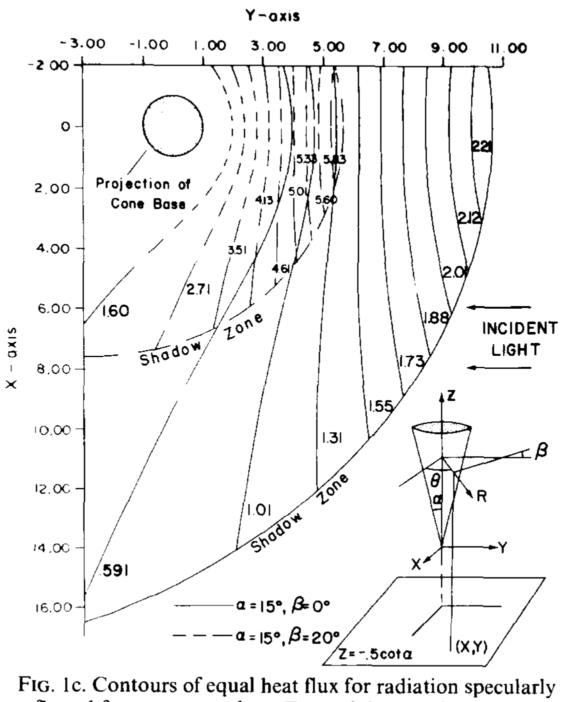

FIG. 1c. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = -0{\cdot}5\cot\alpha$ where the unit of length is the cone base.

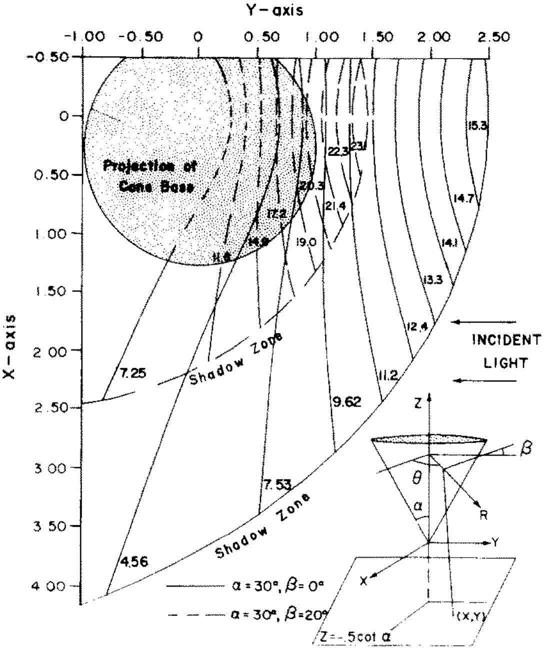

FIG. 1d. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = -0{\cdot}5\cot\alpha$ where the unit of length is the cone base.

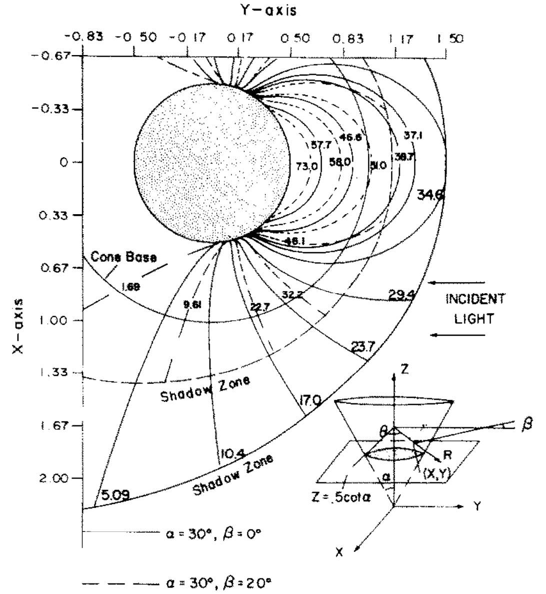

FIG. 1f. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = +0{\cdot}5\cot\alpha$ where the unit of length is the cone base.

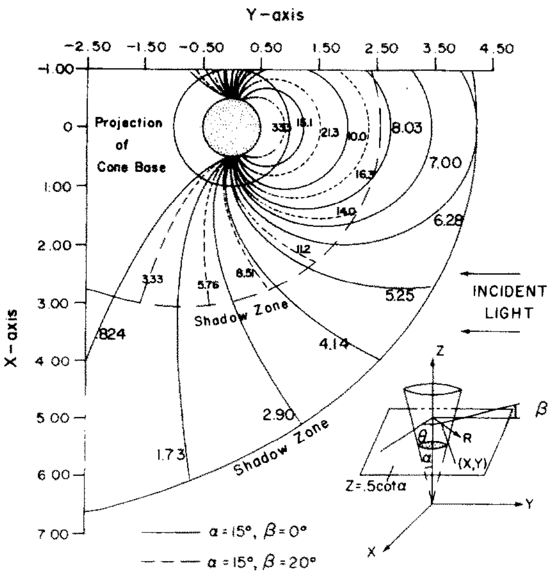

FIG. 1e. Contours of equal heat flux for radiation specularly reflected from cone to plane $Z = +0{\cdot}5\cot\alpha$ where the unit of length is the cone base.

for a more general surface, both f_1 and f_2 will be functions of r and θ. In the present case $\Delta = 1$ and the Jacobian reduces to

$$J = I_0(\theta) + (d - r\cot\alpha)\, I_1(\theta)/r \tag{8}$$

where

$$I_0 = (\cot\alpha\sin\theta - \tan\beta)/(\cos 2\alpha\tan\beta + \sin 2\alpha\sin\theta) \tag{8a}$$

$$I_1 = [2\cos^2\alpha\sin 2\beta\sin\theta - 2\cot\alpha\cos^2\beta\,(\cos^2\alpha\sin^2\theta + \cos^2\theta) - 2\sin 2\alpha\sin^2\beta]/[\cos 2\alpha\sin\beta + \sin 2\alpha\cos\beta\sin\theta]^2. \tag{8b}$$

Combining (8) with the flux flow equation (1), and noting that $\cos\mu = \cos\alpha\cos\beta\sin\theta - \sin\alpha\sin\beta$, $([(\partial z/\partial r)^2 + 1]^{\frac{1}{2}} = \csc\alpha$, and $[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}} = 1$, one obtains the following expression for the flux per unit area on the receiver surface:

$$\mathscr{E} = \frac{s_0 \rho r (\cot\alpha \cos\beta \sin\theta - \sin\beta)}{\pm[r(I_0 - I_1 \cot\alpha) + I_1 Z]} \quad (9)$$

where the "+" or "−" sign is chosen such that the flux density, $\mathscr{E}$, is positive and $Z = d$. Note that if $d = 0$, that is, the receiver plane is $Z = 0$, the flux per unit area on the receiver surface is independent of the radial coordinate of the reflecting point, and as a result, the contours of equal illumination are straight lines as shown in Figs. 1a and 1b. The loci of points on the receiver surface are found by using the equations of the reflected light ray for different values of r and θ. These equations are:

$$X = r\cos\theta + (Z - r\cot\alpha) f_1 \quad (10a)$$

$$Y = r\sin\theta + (Z - r\cot\alpha) f_2 \quad (10b)$$

where f_1 and f_2 are given by equation (7) and (X, Y) are receiving coordinates.

When $d \neq 0$, equation (9) along with (8a) and (8b) is an explicit expression for the flux per unit area on the receiver surface. In order to calculate contours of equal flux on the receiving surface, one assigns a definite value to $(\mathscr{E}/s_0)$ in equation (9) and then solves the three equations (9), (10a) and (10b) simultaneously for the coordinates X, Y on the desired contour to obtain

$$X = \frac{\pm(\mathscr{E}/s_0) I_1 d(\cos\theta - f_1(\theta)\cot\alpha)}{[\rho(\cos\mu/\sin\alpha) \mp (\mathscr{E}/s_0)(I_0 - I_1\cot\alpha)]} + f_1(\theta) d \quad (11a)$$

$$Y = \frac{\pm(\mathscr{E}/s_0) I_1 d(\sin\theta - f_2(\theta)\cot\alpha)}{[\rho(\cos\mu/\sin\alpha) \mp (\mathscr{E}/s_0)(I_0 - I_1\cot\alpha)]} + f_2(\theta) d. \quad (11b)$$

$X(\theta)$, $Y(\theta)$ given by (11a) and (11b) are parametric equations for the coordinates of a given contour (a given value of $\mathscr{E}/s_0$), on the surface $Z = d$. The parameter θ is the polar angle of a reflecting point on the cone. Before assigning values to $\mathscr{E}/s_0$ in (11a) and (11b) one should use equation (9) to calculate typical values for $\mathscr{E}/s_0$ over the region of interest on the receiver plane so that physically possible values for $(\mathscr{E}/s_0)$ are used in (11a) and (11b). Equal flux contours are shown in Figs. 1c–1f for various values of d and α. Flux contours are expressed as per cent of incident radiation density. The reflection coefficient ρ is taken as 1 for all angles of incidence. In practice ρ will depend upon the angle of incidence and polarization of the incident radiation as given by Fresnel's equation. This correction can be taken easily into account [1]. We have employed the Fresnel correction for unpolarized radiation incident on an aluminium cone and find the shift in the value of a contour to range from one or two to several per cent.

A study of the heat flux over a receiver plane truncating a cone was motivated by the need to know the solar flux reflected from the cone and incident upon instrument boxes mounted on the plane. The cone was part of a rocket configuration. With slightly more effort one can calculate the exact flux incident upon rectangular boxes on the plane simply by specifying the coordinates of the corners of the boxes.

We now modify the previous results to allow the receiver surface to be the general plane $Z = D - AX - BY$ instead of the particular plane $Z = d$, the appropriate Jacobian, equation (2), is

$$J = [I_0(\theta) + (D - AX - BY - r\cot\alpha) I_1(\theta)/r]/\Delta \quad (12)$$

where I_0, I_1 are given by (8a) and (8b). And now $\Delta(\theta) = 1 + Af_1(\theta) + Bf_2(\theta)$ where f_1, f_2 are given by (7). The flux flow equation (1) is changed to

$$\mathscr{E} = \frac{s_0 \rho r(\cot\alpha\cos\beta\sin\theta - \sin\beta)\Delta(\theta)}{\pm[r(I_0 - I_1\cot\alpha) + (D - AX - BY)I_1][A^2 + B^2 + 1]^{\frac{1}{2}}} \tag{13}$$

where the "+" or "−" sign is chosen such that the flux is positive. Replacing Z in (10a) and (10b) by $D - AX - BY$ and then eliminating r from (10a) and (10b) and (13) gives an expression for X and Y in terms of θ.

$$X(\theta) = h_1(\theta)D/(1 + h_1(\theta)A + h_2(\theta)B) \tag{14a}$$

$$Y(\theta) = h_2(\theta)D/(1 + h_1(\theta)A + h_2(\theta)B) \tag{14b}$$

$$\mathscr{E} = \frac{s_0 \rho r(\cot\alpha\cos\beta\sin\theta - \sin\beta)|\Delta|}{\pm[r(I_0 - I_1\cot\alpha) + Z(X, Y)I_1][(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}} \tag{16}$$

where

$$h_1(\theta) = f_1 \pm I_1(\mathscr{E}/s_0)(\cos\theta - f_1\cot\alpha)(A^2 + B^2 + 1)^{\frac{1}{2}}/g(\theta)$$

$$h_2(\theta) = f_2 \pm I_1(\mathscr{E}/s_0)(\sin\theta - f_2\cot\alpha)(A^2 + B^2 + 1)^{\frac{1}{2}}/g(\theta)$$

and

$$g(\theta) = \rho\Delta(\theta)(\cos\mu/\sin\alpha) \pm (\mathscr{E}/s_0)(I_0 - I_1\cot\alpha)(A^2 + B^2 + 1)^{\frac{1}{2}}.$$

If the upper sign is taken in (13) then the upper sign applies in h_1 and h_2. Likewise for the lower.

III. SPECULAR REFLECTION FROM CONE TO NON-PLANAR RECEIVER

In the preceding section we have described a procedure for calculating contours of equal illumination when the incident radiation is specularly reflected from a cone to a plane. We shall now describe how this procedure is modified when the receiver surface is nonplanar. The functions f_1 and f_2 are still given by (7); whereas, the Jacobian from equation (2) takes the form

$$J(R, \theta) = [I_0 + (Z(X, Y) - r\cot\alpha)I_1/r]/[1 - f_1(\partial Z/\partial X) - f_2(\partial Z/\partial Y] \tag{15}$$

where I_0 and I_1 are given by equations (8a) and (8b). Note that the numerator of equation (15) is of the same form as the Jacobian obtained for specular reflection from a cone to a plane, equation (8). The only difference is in the functional form of the equation of the receiver surface $Z = Z(X, Y)$. Combining equation (15) with equation (1), one obtains the following expression for the flux per unit area on the general curved receiver surface:

where $\Delta = 1 - f_1(\partial Z/\partial X) - f_2(\partial X/\partial Y)$ and the "+" or "−" sign is chosen such that the flux is positive. The procedure for calculating contours of constant illumination on the receiver surface has been previously described. One assigns a definite value to $(\mathscr{E}/s_0)$ in equation (16) and then solves equation (16) and the two equations (3a) and (3b) representing the reflected light ray which become

$$X = r\cos\theta + (Z(X, Y) - r\cot\alpha)f_1 \tag{17a}$$

$$Y = r\sin\theta + (Z(X, Y) - r\cot\alpha)f_2 \tag{17b}$$

simultaneously for the receiver coordinates X, Y. For most non-planar surfaces it will be necessary to solve equations (16), (17a) and (17b) numerically for X, Y by using, for example, Newton's iteration procedure for solving non-linear systems of equations [2].

IV. SPECULAR REFLECTION FROM A PARABOLOID TO A PLANAR RECEIVER

In this section we shall consider the reflecting surface to be the paraboloid $z = R_0 R^2 \cot\alpha$ where R is the radius of the paraboloid measured from the z axis in units of R_0. R_0 is the radius of the base which we shall consider to be unity and α is the angle between the z-axis and a line drawn from the origin to the base of the paraboloid. The receiver surface will be considered to be the plane $Z = d$ where d is a constant, and the direction of the incident parallel radiation is given by

$$\boldsymbol{i} = -\cos\beta \boldsymbol{J} - \sin\beta \boldsymbol{K}. \tag{18a}$$

The outward unit normal to the reflecting surface is given by:

$$\boldsymbol{n} = (2R\cot\alpha\cos\theta\,\boldsymbol{I} + 2R\cot\alpha\sin\theta\boldsymbol{J} - \boldsymbol{K})/(1 + 4R^2\cot^2\alpha)^{\frac{1}{2}}. \tag{18b}$$

$f_1(\equiv i'_x/i'_z)$ and $f_2(\equiv i'_y/i'_z)$ are given by

$$f_1 = \frac{AR^2\sin 2\theta - BR\cos\theta}{\sin\beta - CR^2 - DR\sin\theta};$$
$$f_2 = \frac{-AR^2\cos 2\theta - BR\sin\theta - \cos\beta}{\sin\beta - CR^2 - DR\sin\theta} \tag{18c}$$

where $A = 4\cot^2\alpha\cos\beta$; $B = 4\cot\alpha\sin\beta$; $C = 4\cot^2\alpha\sin\beta$; and $D = 4\cot\alpha\cos\beta$. After performing the appropriate partial differentiation, the expression for the Jacobian given by equation (2) becomes:

$$J(R,\theta) = I_0(R,\theta) + (Z - z)I_1(R,\theta) + (Z - z)^2 I_2(R,\theta) \tag{19}$$

where

$$I_0 = [(1 + 4R^2\cot^2\alpha)(\sin\beta - 2R\cot\alpha\cos\beta\sin\theta)]/I_z \tag{19a}$$

$$I_1 = 4\cot\alpha[2R\cot\alpha\sin 2\beta\sin\theta \times (1 + 4R^2\cot^2\alpha) - 8R^2\cot^2\alpha \times (1 + 2R^2\cot^2\alpha\cos^2\beta) - \sin^2\beta - 1]/I_z^2 \tag{19b}$$

$$I_2 = 16\cot^2\alpha[(1 + 4R^2\cot^2\alpha) \times (\sin\beta - 2R\cot\alpha\cos\beta\sin\theta)]/I_z^3 \tag{19c}$$

with the abbreviation $I_z = \sin\beta(1 - 4R^2\cot^2\alpha) - 4R\cot\alpha\cos\beta\sin\theta$. Combining (19) with the flux flow equation (1) and using $\cos\mu = (2R\cot\alpha\cos\beta\sin\theta - \sin\beta)/(1 + 4R^2\cot^2\alpha)^{\frac{1}{2}}$; $[(\partial z/\partial R)^2 + 1]^{\frac{1}{2}} = (1 + 4R^2\cot^2\alpha)^{\frac{1}{2}}$ and $[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}} = 1$, one obtains

$$\mathscr{E} = \frac{s_0\rho(2R\cot\alpha\cos\beta\sin\theta - \sin\beta)}{\pm[I_0 + (Z - z)I_1 + (Z - z)^2 I_2]} \tag{20}$$

where "+" or "−" sign is chosen so that flux is positive. In principle one can now calculate, as described in section II, flux density contours. However, the algebra is more involved. Assigning a definite value to $(\mathscr{E}/s_0)$, one obtains a polynomial in R by rearranging (20) with the aid of FORMAC* in the form:

$$\sum_{i=0}^{8} a_i R^i = 0 \tag{21}$$

where the coefficients a_i are given in Appendix 1.

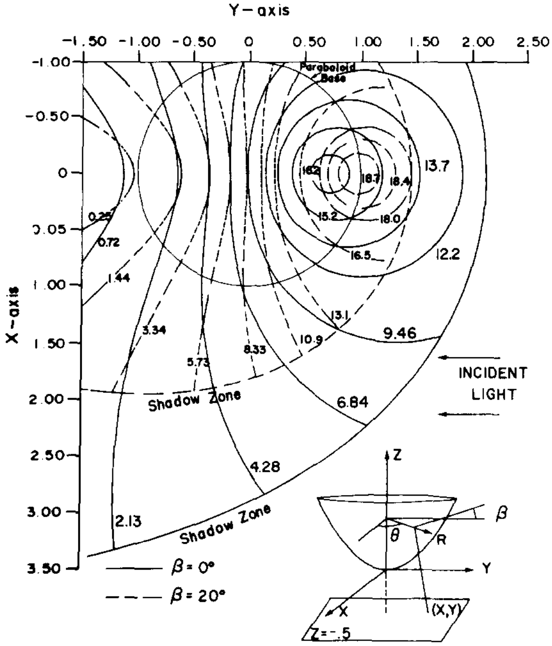

FIG. 2. Contours of equal illumination for light specularly reflected from paraboloid to plane $Z = -0{\cdot}5$ in units of the base radius of the paraboloid. Flux values associated with contours represents per cent of incident radiation. Reflection coefficient is regarded as constant and equal to 1.

* FORMAC Interpreter is an extension of the OS/360 PL/I(F) compiler which provides for the symbolic manipulation of mathematical expressions [3].

The problem is to solve (20) for R as a function of θ and substitute the result into (3a) and (3b) to obtain the coordinates X, Y of the desired contour. Since an analytical expression for R is not possible in this case, the real and imaginary roots of equation (21) were obtained for given values of α, β, $(\mathscr{E}/s_0)$, $Z(\equiv d)$ and θ by using the Newton–Ralphson iterative techniques as described in the IBM Scientific Subroutine Package Program POLRT [4]. The real and physically possible roots were then used to determine the receiver coordinates X, Y for a series of values of θ. Typical constant heat flux contours for $\alpha = 45$ degress and $\beta = 0, 20$ degrees are shown in Fig. 2. The flux values are expressed as per cent of incident radiation flux density. The reflection coefficient is regarded as constant and equal to 1.

V. SPECULAR REFLECTION FROM HEMISPHERE TO PLANAR RECEIVER

Now we shall consider the reflecting surface to be the hemisphere $z = R_0(1 - (1 - r^2/R_0^2)^{\frac{1}{2}})$ where r is the radius of hemisphere measured from the Z axis and R_0 is radius of base which we shall consider to be unity. The receiver surface is again the plane $Z = d$ where d is a constant. For convenience, we put $R = r/R_0$. The incident radiation is given by

$$\boldsymbol{i} = -\cos\beta \boldsymbol{J} - \sin\beta \boldsymbol{K}. \tag{22a}$$

The outward normal to the reflecting surface is

$$\boldsymbol{n} = R\cos\theta \boldsymbol{I} + R\sin\theta \boldsymbol{J} - (1 - R^2)^{\frac{1}{2}}\boldsymbol{K}. \tag{22b}$$

The direction of reflected ray is determined from (4). Then $f_1(\equiv i'_x/i'_z)$ and $f_2(\equiv i'_y/i'_z)$ are given by

$$f_1 = \frac{CR^2 \sin 2\theta - 2AR(1 - R^2)^{\frac{1}{2}}\cos\theta}{-2CR(1 - R^2)^{\frac{1}{2}}\sin\theta + A(1 - 2R^2)}$$
$$f_2 = \frac{2CR^2\sin^2\theta - 2AR(1 - R^2)^{\frac{1}{2}}\sin\theta - C}{-2CR(1 - R^2)^{\frac{1}{2}}\sin\theta + A(1 - 2R^2)} \tag{22c}$$

where $A = \sin\beta$ and $C = \cos\beta$. After performing the appropriate partial differentiation, the expression for the Jacobian given by equation (2) becomes.

$$J(R,\theta) = I_0(R,\theta) + (Z - z)I_1(R,\theta) + (Z - z)^2 I_2(R,\theta) \tag{23}$$

where

$$I_0 = (\sin\beta - R(1 - R^2)^{-\frac{1}{2}}\cos\beta\sin\theta)/I_z \tag{23a}$$

$$I_1 = \{2R\sin 2\beta\sin\theta + 2(1 - R^2)^{-\frac{1}{2}} \times [R^2(\sin^2\beta - \cos^2\beta\sin^2\theta) - \sin^2\beta - 1]\}/I_z^2 \tag{23b}$$

$$I_2 = \{2R\sin 2\beta\sin\theta(1 - R^2)^{-\frac{1}{2}} \times [-3 + 4R^2 + R^2\cos^2\beta\sin 2\theta(1 - R^2)^2] + 4\sin^2\beta(1 - 2R^2) + 8R^2\cos^2\beta\sin^2\theta \times [\sin^2\theta + R^2\cos^2\theta \times (1 + 2\sin^2\beta \times (1 + R_2))]\}/I_z^4 \tag{23c}$$

with the abbreviation $I_z = (1 - 2R^2)\sin\beta - 2R(1 - R^2)^{-\frac{1}{2}}\cos\beta\cos\theta$. Combining (23) with the flux flow equation (1) and using $\cos\mu = R\cos\beta\sin\theta - \sin\beta(1 - R^2)^{\frac{1}{2}}$, $[(\partial Z/\partial R)^2 + 1]^{\frac{1}{2}} = (1 - R^2)^{-\frac{1}{2}}$ and $[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}} = 1$, yields the following expression for the flux per unit area of receiver surface

$$\mathscr{E} = \frac{s_0\rho[R\cos\beta\sin\theta(1 - R^2)^{-\frac{1}{2}} - \sin\beta]}{\pm[I_0 + (Z - z)I_1 + (Z - z)^2 I_2]} \tag{24}$$

where the "+" or "−" sign is chosen so that the flux is positive. The flux flow equation (24) for specified $(\mathscr{E}/s_0)$ is again rearranged into a more manageable polynomial expression by means of the FORMAC program to yield:

$$\sum_{i=0}^{9} a_i R^i + (1 - R^2)^{\frac{1}{2}} \sum_{j=0}^{8} b_i R^i = 0 \tag{25}$$

where the coefficients a_i, b_i are given in Appendix 2.

The flux flow equation in the form (25) can now be solved simultaneously with the equation for the coordinates X, Y to obtain the loci of points of constant flux on the receiver plane. The angle θ again plays the role of a parameter. Typical constant illumination (constant heat flux) contours for $\beta = 0$ and 20 degrees are presented in Fig. 3. Contours are expressed as

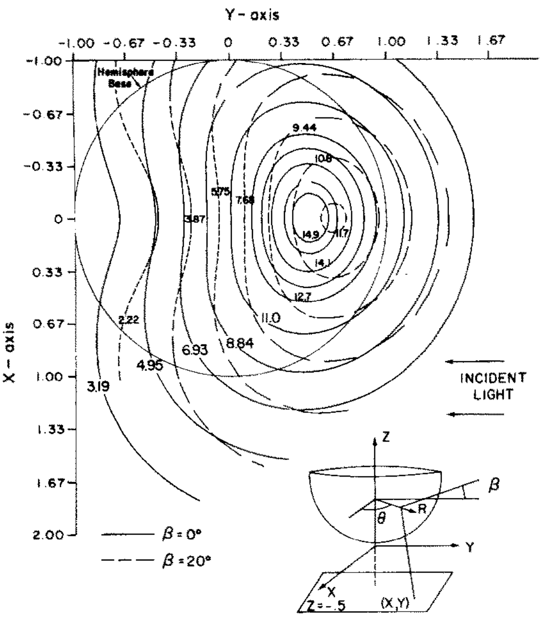

FIG. 3. Contours of equal heat flux for radiation specularly reflected from hemisphere to plane $Z = -0{\cdot}5$ in units of the radius of the hemisphere. Flux values associated with contours represents per cent of incident radiation. Reflection coefficient ρ is one for all angles of incidence.

percentage of incident flux density; reflection coefficient is taken as 1 for all angles of incidence.

CONCLUSION

A simple, exact analytical procedure has been developed which enables one to calculate the flux density over an arbitrary receiver surface for incident radiation reflected from an arbitrary curved surface. As examples, flux contours are calculated over a planar receiver surface for parallel radiation from a cone, a paraboloid and a hemisphere. With equal ease one can calculate flux contours over a curved receiver surface. The procedure should enable one to obtain exact solutions to specular radiant heat transfer problems not previously analytically tractable.

REFERENCES

1. D. G. Burkhard, D. L. Shealy and R. U. Sexl, Specular reflection of heat radiation from an arbitrary reflector surface to an arbitrary receiver surface, *Int. J. Heat Mass Transfer* **16**, 271–280 (1973).
2. F. Scheid, *Numerical Analysis*. McGraw-Hill, New York (1968).
3. J. Xenakis, FORMAC symbolic mathematical interpreter, IBM Contributed Program Library, Hawthorne, N.Y. (1969).
4. System/360 Scientific Subroutine Package, IBM, White Plains, N.Y. (1968).

APPENDIX 1

The coefficients a_i appearing in the polynomial expression (21) for the flux flow equation when reflecting surface is paraboloid of revolution are:

$$a_0 = \rho \sin^4 \beta + (\mathscr{E}/s_0)\sin\beta(1 - 4Z\cot\alpha)(\sin^2\beta - 4Z\cot\alpha)$$

$$a_1 = -2\cos\beta\cot\alpha\sin\theta\{\sin^3\beta[7 + 5(\mathscr{E}/s_0)] + 8Z\sin 2\beta\cot\alpha(\mathscr{E}/s_0) + 16Z^2\cot^2\alpha(\mathscr{E}/s_0)\}$$

$$a_2 = 4\sin\beta\cot^2\alpha\{(1 + 8\cos^2\beta\sin^2\theta)(\mathscr{E}/s_0) + 3\rho\sin\beta(6\cos^2\beta\sin^2\theta - 6\sin^2\beta) - 4Z\cot\alpha(3 - \sin^2\beta + 4\cos^2\beta\sin^2\theta)(\mathscr{E}/s_0) + 16Z^2\cot^2\alpha(\mathscr{E}/s_0)\}$$

$$a_3 = 8\cos\beta\cot^3\alpha\sin\theta\{-(2+3\sin^2\beta+4\cos^2\beta\sin^2\theta)(\mathscr{E}/s_0) + 5\rho\sin\beta(3\sin^2\beta - 4\cos^2\beta\sin^2\theta) + 24Z\cot\alpha(\mathscr{E}/s_0) - 16Z^2\cot^2\alpha(\mathscr{E}/s_0)\}$$

$$a_4 = 16\cot^4\alpha\{\rho(3\sin^4\beta - 6\sin^2(2\beta)\sin^2\theta + 8\sin^4\theta\cos^2\beta) + \sin 2\beta\cos\beta(1 + 4\sin^2\theta)(1 - 2Z\cot\alpha)(\mathscr{E}/s_0)\}$$

$$a_5 = 8\cot^5\theta\sin\theta\{6\sin 2\beta(4\cos^4\beta\sin^2\theta - 3\sin^2\beta) - 4\cos^3\beta(5 + 4\sin^2\theta)(\mathscr{E}/s_0) + 64Z\cos\beta\cot\alpha(\mathscr{E}/s_0)\}$$

$$a_6 = 64\sin\beta\cot^6\alpha\{\rho\sin\beta(6\cos^2\beta\sin^2\theta - \sin^2\beta) + 4\cos^2\beta\cot\alpha\}$$

$$a_7 = 128\cos\beta\cot^7\alpha\sin\theta\{\rho\sin^2\beta - 3\cos^3\beta(\mathscr{E}/s_0)\}$$

$$a_8 = -256\cos^2\beta\sin\beta\cot^8\alpha(\mathscr{E}/s_0).$$

APPENDIX 2

The coefficients a_i, b_j appearing in the polynomial expression (25) for the flux flow equation when the reflecting surface is a hemisphere are:

$$a_0 = 2(\mathscr{E}/s_0)\sin^2\beta(\sin^2\beta - 3)(1 - Z)$$

$$a_1 = 0{\cdot}5\sin 2\beta\sin\theta\{(\mathscr{E}/s_0)[5\sin^2\beta - 4(3Z^2 - 6Z + 4)] - 9\rho\sin^3\beta\}$$

$$a_2 = 2(\mathscr{E}/s_0)(1 - Z)[\sin^2\beta(8 - 5\sin^2\beta) + \cos^2\beta\sin^2\theta(4 - 13\sin^2\beta - 8\sin^2\theta)]$$

$$a_3 = 2\sin 2\beta \sin\theta\{(\mathscr{E}/s_0[0{\cdot}25\sin^2(2\beta) - 4\sin^2\beta - 3 + (4+\cos^2\beta\sin 2\theta) \times (Z^2 - 2Z + 2)] + 2\rho\sin\beta[8\sin^2\beta - 7\cos^2\beta\sin^2\theta]\}$$

$$a_4 = -8(\mathscr{E}/s_0)(1-Z)\{0{\cdot}25\sin^2(2\beta) + \cos^2\beta\sin^2\theta[1+\sin^2\beta(9+4\cos^2\theta) - \sin^2\theta(2+\cos^2\beta) + 2\cos^2\theta]\}$$

$$a_5 = 2(\mathscr{E}/s_0)\sin 2\beta\sin\theta[3\sin^3\beta - \cos 2\beta\sin^2\theta - 2\cos^2\beta\sin 2\theta(5 - 4Z + 2Z^2)] - 8\rho\cos\beta\sin\theta[2\cos^4\beta\sin^4\theta + 21\sin^3\beta - 7{\cdot}75\sin^2(2\beta)\sin^2\theta]$$

$$a_6 = -8(\mathscr{E}/s_0)(1-Z)[\sin^4\beta + \cos^2\beta\sin^2\theta(6\sin^2\beta + \cos^2\beta\sin^2\theta - \cos^2\theta)]$$

$$a_7 = 8(\mathscr{E}/s_0)\sin\beta\cos^3\beta\sin^2\theta\cos\theta(4 - 2Z + Z^2) + 32\rho\cos\beta\sin\theta[6\sin^4\beta - 2{\cdot}75\sin^2(2\beta)\sin^2\theta - \cos^4\beta\sin^4\theta]$$

$$a_8 = 2\sin^2(2\beta)\sin^2(2\theta)(1-Z)$$

$$a_9 = +8\cos\beta\sin\theta[(\mathscr{E}/s_0)\sin\beta\cos^2\beta\sin 2\theta + 2\rho(5\sin^4\beta - 2{\cdot}5\sin^2(2\beta)\sin^2\theta + \cos^4\beta\sin^4\theta)]$$

$$b_0 = (\mathscr{E}/s_0)\sin^2\beta(4Z^2 - 8Z + 6 - \sin^2\beta) + \rho\sin^5\beta$$

$$b_1 = 2(\mathscr{E}/s_0)\sin 2\beta\sin\theta(1-Z)(4 - 3\sin^2\beta)$$

$$b_2 = 4(\mathscr{E}/s_0)\{\sin^2\beta(\sin^2\beta - 2\cos^2\beta\sin^2\theta - 2Z^2 + 4Z - 3) + \cos^2\beta\sin^2\theta[-2 + 4\sin^2\theta(1-Z) + 2Z\sin^2\theta]\} + 8\rho\sin^3\beta(4\cos^2\beta\sin^2\theta - \sin^2\beta)$$

$$b_3 = 4(\mathscr{E}/s_0)\sin 2\beta\sin\theta(1-Z)(5\sin^2\beta - 3\cos^2\beta\sin^2\theta - \cos^2\beta\sin 2\theta - 2)$$

$$b_4 = 4(\mathscr{E}/s_0)\{-\sin^4\beta + \cos^2\beta\sin^2\theta[2 + 3\sin^2\beta - 8\sin^2\theta + 2\cos^2\theta(1 + 2\sin^2\beta) \times (Z^2 - 2Z - 2)]\} + 6\rho\sin\beta[4\sin^4\beta + 8\cos^4\beta\sin^4\theta - 5{\cdot}25\sin^2(2\beta)\sin\theta]$$

$$b_5 = 16(\mathscr{E}/s_0)\sin 2\beta\sin\theta(1-Z)(\cos^2\beta\sin^2\theta - \sin^2\beta - 0{\cdot}5\cos^2\beta\sin 2\theta)$$

$$b_6 = 2(\mathscr{E}/s_0)\cos^2\beta\sin^2(2\theta)[2\sin^2\beta(1-Z)^2 - 1] - 32\sin\beta \times [\sin^4\beta + 4\cos^4\beta\sin^4\theta - 2{\cdot}25\sin^2(2\beta)\sin^2\theta]$$

$$b_7 = -16(\mathscr{E}/s_0)\sin\beta\cos^3\beta\sin^2\theta\cos\theta(1-Z)$$

$$b_8 = (\mathscr{E}/s_0)\sin^2(2\beta)\sin^2(2\theta) + 16\rho\sin\beta[\sin^4\beta + 5\cos^4\beta\sin^4\theta - 2{\cdot}5\sin^2(2\beta)\sin^2\theta]$$

FLUX THERMIQUE SUR UN PLAN POUR UN RAYONNEMENT PARALLELE SPECULAIREMENT REFLECHI PAR UN CONE, UN HEMISPHERE ET UN PARABOLOÏDE

Résumé—Des formules explicites sont établies pour le flux thermique par unité d'aire d'une surface réceptrice générale après qu'un rayonnement incident parallèle soit réfléchi spéculairement par un cône, un paraboloïde de révolution et par un hémisphère. Les formules de la densité de flux sont inversées de telle sorte que les coordonnées des points sur la surface réceptrice sont exprimées en fonction de la densité de flux. Ceci donne des expressions analytiques pour la répartition de la densité de flux. On présente des résultats typiques de répartition constante sur une surface réceptrice plane perpendiculaire à l'axe de symétrie du réflecteur.

LINIEN KONSTANTEN WÄRMESTROMES AUF EINER EBENE FÜR PARALLELE, SPIEGELND REFLEKTIERTE STRAHLUNG VON EINEM KEGEL, EINER HALBKUGEL UND EINEM PARABOLOID

Zusammenfassung—Es werden explizite Gleichungen abgeleitet für die Wärmestromdichte auf einer allgemeinen Empfängeroberfläche, für parallel einfallende, spiegelnd reflektierte Strahlung von einem Kegel, einem Rotationsparaboloid und einer Halbkugel. Die Gleichungen für die Wärmestromdichte werden so aufgelöst, dass die Koordinaten eines Punktes auf der Empfängerfläche von der Stromdichte abhängen. Dies ergibt geschlossene analytische Ausdrücke für die Linien konstanter Wärmestromdichte. Typische Ergebnisse wurden als Linien konstanter Beleuchtung gezeichnet auf einer ebenen Empfängerfläche, die senkrecht zur Symmetrieachse des Reflektors steht.

ЛИНИИ ПОСТОЯННОГО ТЕПЛОВОГО ПОТОКА НА ПЛАСТИНЕ ПРИ ЗЕРКАЛЬНОМ ОТРАЖЕНИИ ПАРАЛЛЕЛЬНОГО ИЗЛУЧЕНИЯ ОТ КОНУСА, ПОЛУСФЕРЫ И ПАРАБОЛОИДА

Аннотация—Получены формулы в явном виде для удельного теплового потока на единицу площади поверхности приемника при зеркальном отражении плоского падающего излучения конусом, параболоидом вращения и полусферой. Формулы для плотности потока преобразованы таким образом, что координаты точек поверхности приемника выражены черeъ плотность потока. Это позволило получить замкнутые аналитические выражения для линий постоянной плотности потока. Типичные результаты представлены на прафиках в виде линий постоянной освещенности на плоской поверхности приемника, перпендикулярной оси симметрии отражателя.